Exercise 6

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

$$y'' + xy = 0, \; y(0) = 0, \; y'(1) = 0$$

Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned} \int_0^x y''(t)\,dt &= \int_0^x u(t)\,dt \\ y'(x) - y'(0) &= \int_0^x u(t)\,dt \\ y'(x) &= y'(0) + \int_0^x u(t)\,dt \end{aligned}$$

In order to determine $y'(0)$, set $x = 1$ in this equation for $y'(x)$.

$$y'(1) = y'(0) + \int_0^1 u(t)\,dt$$

Substitute $y'(1) = 0$ and solve for $y'(0)$.

$$0 = y'(0) + \int_0^1 u(t)\,dt \quad \rightarrow \quad y'(0) = -\int_0^1 u(t)\,dt$$

Substitute this result for $y'(0)$ back into the formula for $y'(x)$.

$$y'(x) = -\int_0^1 u(t)\,dt + \int_0^x u(t)\,dt$$

Integrate both sides from 0 to x again.

$$\begin{aligned} \int_0^x y'(r)\,dr &= \int_0^x \left[-\int_0^1 u(t)\,dt + \int_0^r u(t)\,dt\right] dr \\ y(x) - y(0) &= -x\int_0^1 u(t)\,dt + \int_0^x \int_0^r u(t)\,dt\,dr \end{aligned}$$

Substitute $y(0) = 0$.

$$y(x) = -x\int_0^1 u(t)\,dt + \int_0^x \int_0^r u(t)\,dt\,dr$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^r u(t)\,dt & dw &= dr \\ dv &= u(r)\,dr & w &= r \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} y(x) &= -x\int_0^1 u(t)\,dt + r\int_0^r u(t)\,dt\bigg|_0^x - \int_0^x ru(r)\,dr \\ &= -x\int_0^1 u(t)\,dt + x\int_0^x u(t)\,dt - \int_0^x ru(r)\,dr \\ &= -x\int_0^1 u(t)\,dt + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= -x\int_0^1 u(t)\,dt + \int_0^x (x-t)u(t)\,dt \end{aligned} \tag{2}$$

Now plug equations (1) and (2) into the original ODE.

$$y'' + xy = 0 \quad \rightarrow \quad u(x) + x\left[-x\int_0^1 u(t)\,dt + \int_0^x (x-t)u(t)\,dt\right] = 0$$

Expand the left side.

$$u(x) - x^2\int_0^1 u(t)\,dt + x\int_0^x (x-t)u(t)\,dt = 0$$

Solve for $u(x)$.

$$\begin{aligned} u(x) &= x^2\int_0^1 u(t)\,dt - x\int_0^x (x-t)u(t)\,dt \\ &= \int_0^1 x^2u(t)\,dt - \int_0^x x(x-t)u(t)\,dt \\ &= \int_0^x x^2u(t)\,dt + \int_x^1 x^2u(t)\,dt - \int_0^x x(x-t)u(t)\,dt \\ &= \int_0^x [x^2 - x(x-t)]u(t)\,dt + \int_x^1 x^2u(t)\,dt \\ &= \int_0^x xtu(t)\,dt + \int_x^1 x^2u(t)\,dt \end{aligned}$$

Therefore, the equivalent Fredholm integral equation is

$$u(x) = \int_0^1 K(x,t)u(t)\,dt,$$

where

$$K(x,t) = \begin{cases} xt & 0 \le t \le x \\ x^2 & x \le t \le 1 \end{cases}.$$